Some New Results On \(Q(a)\)-Balance Edge-Magic Graphs

S.Vimala,
Assistant Professor, Department of Mathematics
Mother Teresa Women’s University, Kodaikanal

Abstract: A graph magic if the edges can be labeled with nonnegative real numbers such that (i) different edges have distinct labels, and (ii) the sum of the labels of edges incident to each vertex is the same. This paper discusses \(Q(a)\)- balance magic labeling (BEM) of wheel graph of order \(n \geq 8\), complete graph of order \(n \geq 10\), fan graph of order \(n > 9\) and partition of vertices of a graph.

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1. Introduction

Magic graphs are related to the well-known magic squares, but are not quite as famous. Magic squares are an arrangement of numbers in a square in such a way that the sum of the rows, columns and diagonals is always the same. These mathematical puzzles are known to people for more than 4000 years, but no sooner than in 1960s it was tried to apply into graphs. All graphs in this paper are connected, (multi-)graphs without loop. The graph \(G\) has vertex – set \(V(G)\) and edge – set \(E(G)\). A labeling (or valuation) of a graph is a map that carries graph elements to numbers (usually to the positive or non-negative integers). Edge magic graph introduced by Sin Min Lee, Eric Seah and S.K Tan in 1992[9]. Various author discussed in edge magic graphs like Edge magic\((p,3p-1)\)-graphs, Zykov sums of graphs, cubic multigraphs, Edge-magicness of the composition of a cycle with a null graph\([1,2,6,3]\). In 2007 Sin-Min Lee and Thomas Wong and Sheng-Ping Bill Lo introduced two types of magic labeling on the \(Q(a)\)-Balance Edge-magic Graphs and \(Q(a)\)- Balance Super Edge –magic Graphs and proved several conjectures[12]. The labeling to be edge-magic if the sum of all labels associated with an edge equals a constant independent of the choice of edge, and vertex-magic if the same property holds for vertices.

In this paper, magic labeling, which strengthens in wheel graph, minimum and maximum number of constants in wheel with odd order \((n \leq 8)\) and strong BEM of wheel for \((n \geq 10)\) and BEM of fan graph and complete graph and few more results has to listed.

2. Preliminaries:

A finite graph \(G = [V(G), E(G)]\) without loops, multiple edges or isolated vertices. If there exists a mapping \(f\) from the set of edges \(E(G)\) into positive real numbers such that

(i) \(f(e_1) \neq f(e_2)\) for all \(e_1 \neq e_2 : e_1, e_2 \in E(G)\),

(ii) \(\sum_{e \in E(G)} \eta(v, e)f(e) = r\) for all \(v \in E(G)\) where \(\eta(v, e)\) is 1 when vertex \(v\) and edge \(e\) are incident and 0 in the opposite case.

Then the graph \(G\) is called magic. The mapping \(f\) is called a labeling of \(G\) and the value \(r\) is the index of the label \(f\).

A graph \(G\) is called semimagic if there exists a mapping \(f\) into positive real number which satisfy only the above definition condition (ii)

A graph \(G\) is called edge-magic if there exists a bijective function \(\varphi: V(G) \cup E(G) \rightarrow \{1,2,3,\ldots, |V(G)| \cup |E(G)|\}\) such that \(\varphi(x) + \varphi(xy) + \varphi(y)\) is a constant \(c(\varphi)\) for every edge \(xy \in E(G)\): here \(c(\varphi)\) is called the valence of \(\varphi\).

Unless, A Graph \((p,q)\) –graph \(G\) such that the edges are labeled \(1,2,3,\ldots,p\) so that the vertex sums are constant, mod \(p\), is called edge-magic.

A graph \(G\) is called super edge-magic if \(\varphi(V(G)) = \{1,2,3,\ldots, |V(G)|\}\).

If we require the vertex sums is a constant (mod \(p\)), the graph is called \(Q(a)\)- balance edge-magic (BEM).
For $a \geq 1$, we denote

$$Q(a) = \{\pm a, \ldots, \pm (a-1 +q/2)\}, \text{ if } q \text{ is even,}$$

$$Q(a) = \{0, \pm a, \ldots, \pm (a-1 +(q-1)/2)\}, \text{ if } q \text{ is odd.}$$


3. Main Results (Proved Open Problems in [13])

Prove that $m(W_{2n+1}) = 14n + 11$

**Proof:** Let $W_{2n+1}$ be $v_1v_2 \ldots v_{2n+1}v_1$. Then the following labeling $f$ is a magic labeling as in of $W_{2n+1}$.

$$f(v_{2i+1}) = \begin{cases} 3n+4i+1 & 0 \leq i \leq 1 \end{cases}$$

$$f(v_{3+n}) = \begin{cases} 1 & (v4+n) = 2 \end{cases}$$

$$f(v_{i+2}) = \begin{cases} 6n - 7i & 0 \leq i \leq 1 \end{cases}$$

$$f(v_{i+1} v_{i+2}) = \begin{cases} 5n + 3 + \frac{2i}{n+1} & i = 0, 3 \\ 2n(2i) & i = 1, 2 \end{cases}$$

$$f(v_{n+3} v_{n-1}) = \begin{cases} 5n + i & i = 0, 4 \\ 4n - 2i & i = 1, 2 \\ \frac{3n}{2} & i = 3 \end{cases}$$

For example: fig 3.1 shows that a magic labeling of $W_5$

![fig 3.1](image)

Thus

$$m(W_{2n+1}) \leq 14n + 11$$

if $f$ is a magic labeling of $W_{2n+1}$ then

$$c(f) = \sum f(v) + \sum f(e)$$

$$= \sum f(v) + \sum f(e) + \sum f(v)$$

$$= 1 + 2 + 3 + \ldots + (6n + 4) + \sum f(v)$$

$$= (6n + 4) \frac{(6n+5)}{2} + (2n + 1)$$

$$= 18 n^2 + 15n + 12n + 10 + 2n + 1 = 14n + 11.$$
Similarly prove that \( m(W_n) = 5n + 6 \).

Here to prove that \( m(W_n) \leq 5n + 6 \). The labels 1, 2, 3, …, 3n+1 to label the vertices and the remaining labels 1, 2, …, (2n+3) are used to label the edges so that the sum edge is 5n+6. For example figure 3.2 shows that the magic strength of \( W_n \).

**Theorem 2:** For any even natural number \( n \leq 8 \)

\[
\min(EMP(W_n)) = \frac{n}{2} + (2n + 3) - 1
\]

**Proof:** Let \( m \) be the natural number such that \( n = 2m \)

Let \( v_0 \) be the hub vertex and \( (v_1, \ldots, v_n) \) be the rim vertices ordered counter clockwise.

Set \( f: E(W_n) \to \{1, 2, \ldots, 2n\} \) be the function which admits the rule:

\[
\begin{align*}
    v_1v_{i+1} &= n-3 & i = 1 \\
    v_{i+2}v_{i+3} &= \begin{cases} 
    n + 5 + i & 0 \leq i \leq 1 \\
    n + 6 + 2i & 2 \leq i \leq \frac{n}{2} - 1 \\
    n + 2 + i & \frac{n}{2} \leq i \leq \frac{n}{2} + 2 
    \end{cases} \\
    v_nv_1 &= n+2 \\
    v_0v_{n-i+2n} &= i+1 & 0 \leq i \leq 1 \\
    v_0v_{n-i+2i} &= \frac{n}{2} - i \\
    v_0v_{n-2i} &= (n-2)+2i \\
    v_0v_{n-1} &= n-1 \\
    v_0v_1 &= n+1
\end{align*}
\]

Then:

\[
f(v_0v_k) + f(v_0v_{k+1}) + f(v_kv_{k+1}) = \frac{n}{2} + (2n + 3) - 1
\]
Example: Fig 3.3 shows that minimum value of wheel graph even natural numbers

\[ \text{n=4:} \]

\[ \text{n=6:} \]

\[ \text{n=8:} \]

**Theorem 3:** For any even natural number \( n \leq 8 \)

\[ \max(\text{EMP}(W_n)) = \frac{n}{2} + (3n + 1) \]

**Proof:** Let \( m \) be the natural number such that \( n = 2m \)

Let \( v_0 \) be the hub vertex and \((v_1, \ldots, v_n)\) be the rim vertices ordered counter clockwise.

Set \( f: E(W_n) \rightarrow \{1, 2, \ldots, 2n\} \) be the function which admits the rule:

\[
\begin{align*}
    v_{1+i}v_{2+i} &= \frac{n-4}{2} + 2i \\
    v_{3+i}v_{4+i} &= n-(n-1)+2i \\
    v_{5+i}v_{6+i} &= (n+4)-5i \\
    v_{t+6}v_{t+7} &= (n-4)-5i \\
    v_0v_n &= n+2 \\
    v_0v_{n-1} &= 2n-2 \\
    v_0v_{t+1} &= 2n-3-2i \\
    v_0v_{t+3} &= n+3+5i \\
    v_0v_{t+5} &= n+1-i
\end{align*}
\]

Then : \[ 1 \leq k \leq n \]
Example: fig 3.4 shows that minimum value of wheel graph even natural numbers

n = 4:

\[ f(v_0v_1) + f(v_0v_{k+1}) + f(v_{k}v_{k+1}) = \frac{n}{2} + (3n + 1) \]

n = 6:

\[ \text{fig 3.4} \]

n = 8:

\[ \text{fig 3.4} \]

**Theorem 5.3:** The Wheel W_n is strong BEM for n \( \geq 10 \) (Q(a) – Balance Edge Magic)

**Proof:** If n = 10

It suffices to show that W_{10} is Q(a)- Balanced Edge Magic for a = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

For a \( \geq 1 \) we denote as in [57, 62]

\[ Q(a) = \begin{cases} \pm a, \ldots, \pm (a - 1 + \frac{q}{2}) & \text{if } q \text{ is even,} \\ \{0, \pm a, \ldots, \pm (a - 1 + \frac{q-1}{2}) \} & \text{if } q \text{ is odd.} \end{cases} \]

Here q is even

fig 3.5 shows that W_{10} is strong BEM

\[ \text{fig 3.5} \]

Q(1) BEM
A Q(2)-BEM labeling for $W_{10}$: {2, -2, 3, -3, 8, -4, 5, 6, -6, 7, 9, 4, -8, -7, -9, 10, -10, -5}
A Q(3)-BEM labeling for $W_{10}$: {3, -3, 4, -4, 5, -5, 6, -6, 7, -7, 8, -8, 9, -9, 10, -10, 11, -11}
A Q(4)-BEM labeling for $W_{10}$: {4, -4, 5, -5, 10, -6, 7, -7, 8, -8, 9, -9, 6, -10, 11, -11, 12, -12}
A Q(5)-BEM labeling for $W_{10}$: {5, -5, -6, 6, 7, -7, 8, -8, 9, -9, 10, -10, 11, -11, 12, -12, 13, -13}
A Q(6)-BEM labeling for $W_{10}$: {6, -6, 7, -7, 12, -8, 9, -9, 10, -10, 11, -11, 12, -12, 13, -13, 14, 14}
A Q(7)-BEM labeling for $W_{10}$: {7, -7, 8, -8, 9, -9, 10, -10, 11, -11, 12, -12, 13, 19, 14, -14, 15, -15}
A Q(8)-BEM labeling for $W_{10}$: {8, -8, 9, -9, 14, -10, 11, -11, 12, -12, 13, -13, 10, -14, 15, -15, 16, 16, 17, -17}
A Q(9)-BEM labeling for $W_{10}$: {9, -9, 14, -10, 11, -11, 12, -12, 13, -13, 10, -14, 15, -15, 16, 16, 17, -17, 18, -18}
A Q(10)-BEM labeling for $W_{10}$: {14, -10, 11, -11, 12, -12, 13, -13, 10, -14, 15, -15, 16, -16, 17, -17, 18, -18}

**Theorem 4:** The Fan graph $F(1,n)$ Strong BEM $n > 8$

**Proof:** If $n = 9$, $F(1,9)$

$$Q(a) = \{ \pm a \ldots \pm (a - 1 + \frac{q}{2}) \} \text{ if } q \text{ is even},$$

$$Q(a) = \{ 0, \pm a \ldots \pm (a - 1 + \frac{q-1}{2}) \} \text{ if } q \text{ is odd}.$$  

Here $q$ is odd:

Fig 3.6 shows that $F_9$ is strong BEM

**fig 3.6**

A Q(2)-BEM labeling for $F(1,9)$: {0, 2, -2, 3, -3, 8, -4, 5, 6, -6, 7, 9, 4, -8, -7, -9, -5}
A Q(3)-BEM labeling for $F(1,9)$: {0, 3, -3, 4, -4, 5, -5, 6, -6, 7, -7, 8, -8, 9, -9, 10, -10}
A Q(4)-BEM labeling for $F(1,9)$: {0, 4, -4, 5, -5, 10, -6, 7, -7, 8, -8, 9, -9, 6, -10, 11, -11}
A Q(5)-BEM labeling for $F(1,9)$: {0, 5, -5, -6, 6, 7, -7, 8, -8, 9, -9, 10, -10, 11, -11, 12, -12}
A Q(6)-BEM labeling for $F(1,9)$: {0, 6, -6, 7, -7, 12, -8, 9, -9, 10, -10, 11, -11, 8, -12, 13, -13}
A Q(7)-BEM labeling for $F(1,9)$: {0, 7, -7, 8, -8, 9, -9, 17, 10, -10, 11, -11, 12, -12, 13, 19, 14, -14}
A Q(8)-BEM labeling for $F(1,9)$: {0, 8, -8, 9, -9, 14, -10, 11, -11, 12, -12, 13, -13, 10, -14, 15, -14, 15, -15}
A Q(9)-BEM labeling for $F(1,9)$: {0, 9, -9, 14, -10, 11, -11, 12, -12, 13, -13, 10, -14, 15, -15, 16, -16}
A Q(10)-BEM labeling for $F(1,9)$: {0, 14, -10, 11, -11, 12, -12, 13, -13, 10, -14, 15, -15, 16, -16, 17, -17}
Theorem: The complete graph $K_n$ is strong BEM for $n \geq 9$.

Proof: If $n = 9$, $K_9$:

$$Q(a) = \{ \pm a, \ldots, \pm (a - 1 + \frac{q}{2}) \text{ if } q \text{ is even,}$$

$$\{0, \pm a, \ldots, \pm (a - 1 + \frac{q-1}{2}) \text{ if } q \text{ is odd.}$$

Here $q$ is odd:

Fig 3.7 shows that $K_9$ is strong BEM

A Q(1)-BEM labeling for $K_9$: \{0, -1, 2, -2, 3, -3, 4, -4, 5, -5, 6, -6, 7, -7, 8, -8, 1, 9 -9, 10, -10, 11, -11, 12, -12, 13, -13, 14, -14, 15, -15, 16, -16, 17, -17, 18 -18\}

A Q(2)-BEM labeling for $K_9$: \{0, 2, -2, 3, -3, 4, -4, 5, -5, 6, -6, 7, -7, 9, -9, 4, -4, -7, -9, -5, 10, -10, 11, -11, 12, -12, 13, -13, 14, -14, 15, -15, 16, -16, 17, -17, 18 -18, 19, -19\}

A Q(3)-BEM labeling for $K_9$: \{0, 3, -3, 4, -4, 5, -5, 6, -6, 7, -7, 8, -8, 9, -9,10, -10,11,-11,12,-12,13,-13,14,-14,15,-15,16,-16,17,-17,18-18,19,-19,20,-20\}

A Q(4)-BEM labeling for $K_9$: \{0, 4, -4, 5, -5, 10, -6, 7, -7, 8, -8, 9, -9, 6, -10,11,-11, 12, -12, 13, -13, 14, -14, 15, -15, 16, -16, 17, -17, 18 -18, 19, -19, 20, -20, 21, -21\}

A Q(5)-BEM labeling for $K_9$: \{0, 5, -5, -6, 6, 7, -7, 8, -8, 9, -9,10, -10, 11, -11, 12, -12, 13, -13, 14, -14, 15, -15, 16, -16, 17, -17, 18 -18, 19, -19, 20, -20, 21, -21, 22,-22\}

A Q(6)-BEM labeling for $K_9$: \{0, 6, -6, 7, -7, 12, -12, 8, -9, 9, -9, 10, -10, 11, -11, 8, -12, 13, -13, 14, -14, 15, -15, 16, -16, 17, -17, 18 -18, 19, -19, 20, -20, 21, -21, 22,-23\}

A Q(7)-BEM labeling for $K_9$: \{0, 7, -7, 8, -8, 9, -9, -17, 10, -10, 11, -11, 12, -12, 13, 19, 14, -14, 15, -15, 16, -16, 17, -17, 18 -18, 19, -19, 20, -20, 21, -21, 22,-22,23,-23,24,-24\}

A Q(8)-BEM labeling for $K_9$: \{0, 8, -8, 9, -9,14, -10,11, -11, 12, -12, 13, -13, 10, -14,15, -15,16, -16, 17, -17, 18 -18, 19, -19, 20, -20, 21, -21, 22,-22,23,-23,24,-24,25,-25\}

A Q(9)-BEM labeling for $K_9$: \{0, 9, -9, 14, -10, 11, -11, 12, -12, 13, -13, 10, -14,15, -15,16, -16, 17, -17, 18 -18, 19, -19, 20, -20, 21, -21, 22,-22,23,-23,24,-24,25,-25,26,-26\}

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