A Mathematical Analysis for Options Outside Europe: The case of Asian

Marco Mele
University of International Studies, Rome

Abstract
In this paper we want analyze some pricing models for options Asian type and appropriate perturbation methods that allow it to find satisfactory approximations of prices and implied volatility with respect in these models. The class of models which we will go to fill the gaps in the Monte Carlo method. In particular, for estimation of the volatility will be necessary that it be a function of time and of the underlying, or that it proves described as (St,t). The model that we have built over to estimate volatility and its rupture, also examine the best price and the constant elasticity of variance model and this will allow us to obtain the areas of the implied volatility. In this way we will approach estimates that most represent those of the market, though not entirely satisfactory. So, given a fixed maturity, the implied volatility curves resulting from the equations will always decreasing with respect to the strike.

With this model, therefore, you can reproduce a wide range of areas of volatility as it will be possible to act on n, calibration of parameters in order to reproduce as efficiently as possible a series of market prices of an option Asian, compared to those that would result from a Monte Carlo models.

Key words: Asian Option, Forex Market, Monte Carlo Model, CEV
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1. Introduction: about Asian options
A derivative (derivative security) is a financial contract whose value depends on the evolution of the value of another asset (or most other goods), which is called the underlying asset (underlying security) of the title derivative; it is often said that the title is written on the underlying derivative. Typical examples of contracts derivatives are financial options, forward contracts, futures, but also indexed bond contracts and swaps. They are also written derivative contracts on a variety of underlying assets, such as equities, bonds, foreign currencies or currency exchange rates, but also on various goods. The most important example of a derivative contract is specified by the financial, in its basic form (plain vanilla), is a contract that gives the holder the right but not the obligation to buy or sell a predetermined quantities to the title underlying asset at a fixed price contract and that the exercise price (strike price) of the option. The option confers the right to buy is called type of call option, one that provides the right to sell is called a put option type. Normally the standard options provide an expiration date, after which the holder loses the right not exercised to purchase or sell. An important distinction concerns the role of the expiration date if the option can be exercised only on that date is called European option, and whether exercise can be done at any time before the expiration date, we refer to American type. As often happens in finance, the terminology has nothing to do with the limitations of geography: in both the U.S. and Europe are traded options of both types. In the case of European options the expiration of the option is also called the exercise date.

In this context, we can introduce those which are the Asian options. They (Asian or average-rate options) belong to the family path dependent options (path dependent) as their product depends on an average value calculated on the basis of prices that relate to a predetermined set of observations. Negotiated for the first time in Tokyo at the end of the 80s, the Asian options are overwhelmingly revenue in financial markets both as a financial asset in its own right, both as a clause incorporated in the regulation of some financial stocks. The emerging demand for options trading in Asia stems from the fact that these options allow you to reduce the possibility of manipulating the prices of the underlying assets corresponding to the time periods prior to the expiry of the options. The presence of an average value which fundamental parameter reference for the calculation of the product of Asian
options permit leveling price volatility. The assets underlying the Asian options are typically petroleum products, precious metals, foreign currencies, and all those goods subject to negotiations is not very frequent but substantial equivalent. Unlike what happens for the ordinary options, which are normally issued with a maturity of 3, 6 or 9 months, the duration of Asian options can reach 3 years. The calculation of the average price, which acts as fundamental parameter of reference for the determination of the product of Asian options can be influenced by several factors, including: i) the time period used for the calculation of the average; ii) the type of average; iii) the type of sampling. The first is achieved when the Asian option is issued at time 0 = 0 and ends at t = T, the time used to calculate the average can be a subset of [0; T] or can begin even before the time t0 = 0. Usually the last data used for the calculation of the average ST, ie the price of the asset underlying the option observed at the time t = T. The second, however, when in real cases, reference is made above the arithmetic mean; sometimes, for theoretical reasons, it is useful to consider also the geometric mean and The third if the price of the asset can be determined by sampling discrete or continuous. In the first case, the average is calculated according to a defined set of prices S(tk); k = 0; 1; 2; ... n, in the second case is performed with continuity instant by moment; in our case the detection was with discrete sampling. In this case we define \( t_0 = 0; t_n = T \) the reference time period; the arithmetic mean price \( S_n \) in the discrete case is: \( S_n = \frac{1}{n} \sum_{k=1}^{n} S(t_k) \) where \( t_k = t_0 + h k \) and (k = 0; 1; ... n; h = (t_n - t_0)/n).

In the context of Asian options you can distinguish the average options price and average strike options. The first state that the average price, calculated to one of the previous reports, is regarded as the price final average, the latter predict that this average carries the strike price. The pay-off or the amount of money you have to shell out the one who gives option rights, the option expires average price is: \( (1/n \sum_{k=1}^{n} S_k - K) = \max (1/n \sum_{k=1}^{n} S_k - K, 0) \) where K is the exercise price.

2. Hedging strategies and investment

Option contracts are contracts that allow to meet the needs of investor more risk averse investor as well as more sophisticated. These technologies enable the take short positions or higher, limiting the risk assumed, but not the potential profits. In addition, it can operate with respect to a new variable: volatility. We see these different aspects in detail. Most attractive feature of the options is represented by insurance (or coverage or hedging) that they provide. For example, consider an investor who owns a portfolio diversified actions and decides to hedge the downside risk by purchasing put options stock index (this strategy is known as portfolio insurance strategy). In this case the investor benefits from a rising market because it has action and loses up to a maximum equal to the premium paid if the stock market goes down. If you wanted to make a coverage futures through the result would be the sterilization of both earnings losses. in this case, however, the strategy does not entail any loss as a priori no premium should be paid.

The options can also be used to take advantage of price movements to a risk limited. In this case, the investor creates exposure, for example with a Call option, characterized only by the outlay of the premium and the potential gain if the market goes up. This is essentially speculative use of the option contracts. The exposure you get in this case, however, is different from that which the speculator would have if it acquired the underlying asset. In first case, he is able to make a potential speculative gain (which is realized if the market goes up), and if the predictions are wrong, your maximum loss is limited to the premium paid. This does not happen if the speculator buys the underlying asset directly. However the option is not most appropriate tool to make speculations as its price reacts less to changes of the underlying compared for example to a contract futures and involves an initial outlay: the award. The leverage and greater with a futures contract, therefore a pure speculator should operate in futures and in options.

In order to reduce the initial outlay, but nonetheless guaranteeing coverage on losses, one speculator or a hedger can also choose strategies given particular combinations of option contracts. These strategies are for example the bull spread where the vision of the market a slight rise, and bear spreads instead if you bet on a slight downward market. The first is the purchase (sale) of a call to an exercise price lower than the purchase and sale (purchase) of a call to an exercise price high, but with the same maturity.
Operators are particularly refined options allow (the only instrument in the panorama of derivatives) to speculate on market volatility. A speculator who expects a period volatility more higher than that generally expected, even if no conviction on the direction of market, it can simultaneously buy a put and a call. This is called a long position straddle, depending on whether the exercise price of the put or the call whether or not identical. If volatility rises, the investor actually earns. In fact, one of the two options is now in a position of high gain and benefit both in terms of price thanks to greater volatility. If volatility rather not climb the speculator loses the premium of both options.

3. The constant elasticity of variance model or Monte Carlo model for options pricing

According to the constant elasticity of variance model (Coex (1975)), we assume that the dynamics of the underlying stochastic differential equation is determined by the type:
\[ dS_t = rS_t dt + \sigma(t)A(S_t) dW_t \]
and the relative price of the call option with strike \( K \) and maturity \( T \), will be given from:
\[ u(t; s) = \frac{1}{2} \sigma^2 t A^2 (s) \left[ \frac{1}{s} \right] e^{-r(T-t)} \left( S_t - K \right) \]
and \( u(T; s) = (s - k) \) \( s \geq 0 \) which represents a perturbation of the model.

With the constant elasticity of variance model we can reduce the disturbance of the Asian option at through approximation volatility implicit:
\[ \alpha \beta \left( t; s; K; r \right) = \frac{\alpha}{1 - \beta} \left( 1 + \frac{\left( 1 - \beta \right)(2 + \beta)}{\beta} \left( e^{-r(T-t)} \right) \right) \]

In view of the previous writing, it’s could implement this formula with Stata or Mathematics in reference to a call option whose underlying security takes for \( t = 0 \) an initial value \( s = 1 \), using as parameters of the model:
\[ \alpha(t) = \alpha = 0.6; \beta = 1/3 \]
If instead we were to use the Monte Carlo method we should prepare as follows. We set a limited function \( F: [0,1] \) included in an actual field (R) and \( X_n \) is a sequence of v.a. independent uniform in \( [1,0] \).

The succession of v.a. \( \{f(X_n)\} \) will still be formed by v.a. independent, all of average \( E[f(X_1)] \) and for law of large numbers will get:
\[ \frac{1}{n} \sum_{k=1}^{n} f(X_k) \]
and for large n it will tend toward \( E(f(X_1)) \).

These observations suggest a method of numerical calculation about the average and just have a random generator: \( X_1, X_2 \ldots \) on \( [0,1] \) and to calculate:
\[ \frac{1}{n} \sum_{k=1}^{n} f(X_k) \]
and finally, it should provide for the calculation of the confidence interval of the price, which will fall with a probability of 95 per cent in the range.
However, the use of this method, requires suffer the possible fluctuations of the volatility, and in particular the presence of disturbances detected, however, in constant elasticity of variance model. In addition, with the Monte Carlo method is not evident with the behavior that sometimes take the curves of volatility relative to market prices. In fact, it may happen that these curves are initially decreasing and becoming increasing after a low point; hence the term smile, with which are called this type of curves.

Therefore, in the following paragraph, we report a method that allows you to find the following asymptotic approximation of the price of a call option Asian according to constant elasticity of variance model.

4. A constant elasticity of variance model

If at time \( t \) the underlying stock’s worth the price of the option with maturity \( T \) and strike \( J \) shall be given by: 

\[
Z (t; s) = e^{-r(T-t)} u(t; s) \quad \text{where} \quad u(t; s) = E \left[(S_{t}^{s} - K)^{+}\right]
\]

and the process of \( S_{t}^{s} \) will be:

\[
\begin{aligned}
\text{d}S_{t} & = rS_{t} \text{d}t + \sigma(t) A(S_{t}) \text{d}W_{t} \\
S_{t} & = s
\end{aligned}
\]  

[1]

Now suppose we are faced with a function of the type \( Q(s) = s^{\beta} \) and that it is a real new function different from \( A(S_{t}) \) in [1] and therefore \( Q(xy) = Q(x), Q(y) \forall \ x, y > 0. \)

In this situation the price of the call option will approximate as:

\[
Z (t; s) \approx e^{-r(T-t)} \left[ e^{-r(T-t)}(S_{t}^{s} - K) \int_{-\infty}^{\sigma} e^{x^2/2} \text{d}x + \sqrt{\frac{\theta}{2}} e^{-r(T-t)x^{2}/2} \right]
\]

We can observe them as we are a source problem about parabolic type with non-constant coefficients. Seen it’s do not know analytical solutions for this problem, we will use the appropriate changes of variables that allow us approximates the solution.

Therefore we set a function as \( f(t, s) = \delta Q(r(t); x(t; s)) \) trying to check if it solves the problem of not estimate the coefficients. therefore, we have before \( \delta A(r(t); x(t; s)) \) with \( \delta A(0, 1/\sqrt{\delta}) = (s-k)^{+} \) and developing around \( Q \), we get \( Q(k+\alpha \delta) = Q(k) \left[ 1+\delta v_{1}x+1/2\delta v_{2}x^{2} \right] \).

Where \( v_{1} = Q^{1}(K) = Q(K) \) and \( v_{2} = Q^{2}(K) = Q(K) \) and consequently \( Q^{2}(K+\delta) = Q^{2} \left[ 1+2\delta v_{1}x+\delta v_{1}^{2}x^{2}+\delta v_{2}^{2}x^{2} \right] \).

Now we approximate \( A \) by the following \( A^{+}(\phi, x) = A^{0}(\phi, x) + \delta A^{1}(\phi, x) + \delta^{2}A^{2}(\phi, x) \) and then:

\[
\begin{aligned}
A^{0} & \left[ \frac{1}{2} \right] A^{0}_{\phi} = 0 \\
A^{0} & (0, x) = x^{+}
\end{aligned}
\]

\[
\begin{aligned}
A^{1} & \left[ \frac{1}{2} \right] A^{1}_{\phi} = \delta A^{0}_{\phi} \\
A^{1} & (0, x) = v_{1x} A^{0}_{\phi}
\end{aligned}
\]

Therefore:
\[
\begin{align*}
A_0 - \frac{1}{2} A_{xx} = v_1 x_1 A_{xx} + \frac{1}{2} \varepsilon^2 (v_1^2 + v_2) + x^2 A + A^3 f(\emptyset, x)
\end{align*}
\]
\[
A(0, x) = x^t
\]

Where:

\[
f = [-v_1 x A_{xx}^2 + 1/2 (v_1^2 + v_2) x^2 (A_{xx}^2 + 2A_{xx} A^3)]
\]

Approximating \( A \) with \( A^* \) the Asian option pricing formula becomes:

\[
V(t,s) \approx e^{-r(t-t)} Q(K) A^* (\phi(t), e^{-r(t-t)} s - \frac{k}{Q(K)})
\]

The operation previously used is homogeneous. Moreover, even if the function at the initial time will not be limited, it grows more slowly of \( \varepsilon^{1/2} \). Therefore, we generated an operator on the pricing of Asian options by solving the problem of the disruption and volatility of a portfolio of derivative.

5. Conclusion

In our work we wanted analyze the price of Asian options by making a comparison of the effectiveness and efficiency of the two mathematical approaches. We were able to observe how the simulation of the share price and the underlying done through equations about Monte Carlo’ methods is extremely sensitive to the expected return and does not consider the presence of the perturbation in the system. However, we are conscious of not having considered the average yield in this model proves to be a constant parameter of the time series in an options market. The methodology for calculating the Monte Carlo are not in our opinion appropriate and sensitive to volatility because, this method provides a trajectory of prices constructed from the average values simulated in the models. In contrast, the model constant elasticity of variance, offers a better alternative for the analysis of prices in the medium and long term related to Asian options. In particular, since the analytical pricing of Asian options is represented under a geometric Brownian motion, the process is already well established through the approach of randomization or through analysis of exponential approximation. For this purpose to be achieved and analyzed the price in an efficient manner, it was necessary that the process of the price selected \( (Ax = Qx) \) provides an evaluation method valid for Asian options and for its function of transition density.

References


